

HIGHER-SPIN CHERN-SIMONS THEORIES
IN ODD DIMENSIONSJOHAN ENGQUIST¹ AND OLAF HOHM¹¹*Institute for Theoretical Physics and Spinoza Institute, Utrecht University
3508 TD Utrecht, The Netherlands*

ABSTRACT

We construct consistent bosonic higher-spin gauge theories in odd dimensions $D > 3$ based on Chern-Simons forms. The gauge groups are infinite-dimensional higher-spin extensions of the Anti-de Sitter groups $SO(D-1, 2)$. We propose an invariant tensor on these algebras, which is required for the definition of the Chern-Simons action. The latter contains the purely gravitational Chern-Simons theories constructed by Chamseddine, and so the entire theory describes a consistent coupling of higher-spin fields to a particular form of Lovelock gravity. It contains topological as well as non-topological phases. Focusing on $D = 5$ we consider as an example for the latter an $AdS_4 \times S^1$ Kaluza-Klein background. By solving the higher-spin torsion constraints in the case of a spin-3 field, we verify explicitly that the equations of motion reduce in the linearization to the compensator form of the Frønsdal equations on AdS_4 .

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1 INTRODUCTION

The construction of theories describing consistently interacting higher-spin fields is for several reasons of great interest. For one thing string theory contains an infinite tower of massive higher-spin states, and it is an old idea that these hint to a spontaneously broken phase of a theory with a huge hidden gauge symmetry, thus extending the geometrical framework of Einstein's theory [1, 2, 3, 4, 5, 6]. However, the actual formulation of higher-spin theories is usually precluded by the interaction problem. The latter refers to the apparent impossibility of introducing interactions into a free higher-spin (HS) theory in such a way, that the number of dynamical degrees of freedom is unaltered [7, 8]. For instance, naively coupling free massless HS fields to gravity violates the HS gauge symmetry and thus renders the theory inconsistent [9].

In a series of paper Vasiliev has, however, begun to find a route avoiding these no-go theorems, i.e. to consistently couple HS fields to gravity, by relaxing the following assumptions. First, the theory is assumed to have a non-vanishing negative cosmological constant – leading to Anti-de Sitter (*AdS*) instead of Minkowski space as the ‘ground state’ – and to depend on this cosmological constant in a non-polynomial way. The latter excludes a flat space limit, in accordance with standard S-matrix arguments [10]. Second,

it will necessarily contain an infinite number of massless fields carrying arbitrarily high spin, whose couplings can be of arbitrary power in the derivatives (see [11] for a review and references therein).

The formulation of the associated HS theory is based on a gauging of an infinite-dimensional HS algebra, in the same way that gravity and supergravity theories can be viewed as resulting from a gauging of a (super-)AdS algebra. However, theories which are constructed along these lines (as, e.g., in the approach of MacDowell-Mansouri [12]) are not true gauge theories in that the gauge symmetry is not manifest and, moreover, (super-)torsion constraints have to be imposed by hand. For instance, in supergravity invariance under local supersymmetry is by no means manifest and has to be checked explicitly. In addition, so-called ‘extra fields’ appear in HS theories, which are unphysical and have to be expressed in terms of the physical fields by imposing further constraints. In total, the program of Vasiliev consists of finding a non-linear HS theory [11], which

- (i) is still invariant under (a deformation of) the HS symmetry, and
- (ii) yields in the linearization the required free field equations.

Of course, both requirements are related since once it is proven that the free field equations are of 2nd order, the HS symmetry, i.e. (i), fixes the field equations uniquely to the so-called Frønsdal form. In the approach of [13, 14] this requirement is implemented through the condition that the extra fields decouple in the free limit (for reasons we will explain below). However, these conditions have no natural interpretation from the point of view of the HS gauge symmetry. In turn the consistency of the resulting HS action can only be checked up to some order, as it has been done in $D = 4$ and $D = 5$ for cubic couplings [13, 14]. But there are even reasons to expect that this consistency will not extend to all orders [14]. In fact, up to date a fully consistent action describing interactions of propagating HS fields is not known.

An approach, which is instead followed in order to describe consistent HS interactions at the level of the equations of motion, is given by the so-called ‘unfolded formulation’ [15, 16, 17]. The latter is a surprisingly concise way to keep the HS invariance manifest. However, in this approach there is not only an infinite number of physical HS fields, but each of the infinite fields has an infinite number of auxiliary fields, which, roughly speaking, parametrize all space-time derivatives of the physical field. This in turn complicates the analysis of the physical content, and it would be clearly desirable to have a conventional action principle that extends the Einstein-Hilbert action in the same way as supergravity does for spin-3/2 fields.

Concerning the problem of finding a consistent HS action, it should be noted that one example does exist: the Chern-Simons action in $D = 3$ constructed by Blencowe based on a HS algebra [18]. (See also [19, 20] and [21, 22] in a related context.) As the Chern-Simons theory is a true gauge theory, the resulting HS theory is consistent by construction and naturally extends the Einstein-Hilbert action (which in $D = 3$ also has an interpretation as a Chern-Simons action [23].) It is, however, only of limited use since it is topological and does not give rise to propagating degrees of freedom. On the other hand, gauge invariant Chern-Simons actions exist in all odd dimensions, and even though they are topological in any dimension in the sense that they do not depend on a metric, they are not devoid of local dynamics in $D > 3$. In fact, it has been shown

by Chamseddine [24, 25] that the Chern-Simons actions based on the AdS_D algebras $\mathfrak{so}(D-1, 2)$ are equivalent to a particular type of Lovelock gravity with propagating torsion and thus by far not dynamically trivial. So one might wonder what happens if one defines a Chern-Simons action based on a HS extension of $\mathfrak{so}(D-1, 2)$. This paper is devoted to the analysis of this question.

The organization of the paper is as follows. In sec. 2 we briefly review the known free HS theories on Minkowski and AdS , and we introduce the HS Lie algebras which will later on serve as gauge algebras. The general construction of Chern-Simons actions in odd dimensions will be reviewed in sec. 3.1, together with the realization of Lovelock gravity as a Chern-Simons gauge theory. In sec. 3.2 we construct an invariant tensor of the HS algebra, which in turn allows a consistent extension of Chern-Simons gravity to include an infinite tower of HS fields. The constructed theory is then linearized around the ‘non-topological’ Kaluza-Klein background $AdS_4 \times S^1$ in sec. 4. Focusing on the spin-3 mode, we show that the equations of motion reduce to the correct free equations on AdS_4 . We conclude in sec. 5, while technical details concerning Young tableaux, the symmetric invariant and the spin-3 Riemann tensor are relegated to the appendices A–C.

2 HIGHER-SPIN THEORIES AND THEIR GAUGE ALGEBRAS

In this section we first review free HS theories on Minkowski and AdS backgrounds, and then introduce the infinite-dimensional HS Lie algebras, which are the starting point for the construction of interacting HS theories. The results hold in general odd dimensions, though for concreteness we will often specify to $D = 5$.

2.1 FREE HIGHER-SPIN ACTIONS

Bosonic fields of arbitrary spin s are described by symmetric rank- s tensors $h_{\mu_1 \dots \mu_s}$. In the massless case they are subject to the gauge symmetry

$$\delta_\epsilon h_{\mu_1 \dots \mu_s} = \nabla_{(\mu_1} \epsilon_{\mu_2 \dots \mu_s)} , \quad (2.1)$$

parametrized by a symmetric transformation parameter ϵ of rank $s-1$. An action for a free field of spin s on Minkowski and (anti-)de Sitter backgrounds has been given by Frønsdal [28, 29]. For a spin-3 field $h_{\mu\nu\rho}$, which is the case we will later on examine in more detail, it is of the form

$$\begin{aligned} S = \frac{1}{2} \int d^D x [& \nabla_\mu h_{\nu\rho\sigma} \nabla^\mu h^{\nu\rho\sigma} - 3 \nabla_\mu h^\rho{}_{\rho\nu} \nabla^\mu h_\sigma{}^{\sigma\nu} + 3 \cdot 2 \nabla_\mu h^\rho{}_{\rho\nu} \nabla_\sigma h^{\sigma\mu\nu} \\ & - 3 \nabla^\mu h_{\mu\nu\rho} \nabla_\sigma h^{\sigma\nu\rho} - \frac{3}{2} \nabla_\mu h^{\mu\rho}{}_\rho \nabla_\nu h^{\nu\sigma}{}_\sigma + \mathcal{L}_m] . \end{aligned} \quad (2.2)$$

Here ∇_μ denotes the AdS -covariant derivative or a partial derivative in case of a Minkowski background. In the flat case the additional term \mathcal{L}_m vanishes, while on AdS the HS gauge symmetry requires a mass-like term proportional to the cosmological constant. The latter then amounts to the equations of motion

$$\mathcal{F}_{\mu\nu\rho}^{\text{AdS}} \equiv \square h_{\mu\nu\rho} - 3 \nabla_{(\mu} \nabla \cdot h_{\nu\rho)} + 3 \nabla_{(\mu} \nabla_\nu h'_{\rho)} - \frac{1}{L^2} [(D-3) h_{\mu\nu\rho} + 2 \cdot 3 g_{(\mu\nu} h'_{\rho)}] = 0 , \quad (2.3)$$

which defines the so-called Frønsdal operator \mathcal{F} . Here h'_μ denotes the trace of h in the AdS metric and L is the AdS radius, related to the cosmological constant by $L = 1/\sqrt{\Lambda}$. Let us finally note that the given action or field equations are invariant under the gauge variations only if the transformation parameter is traceless and that for spin $s > 3$ a double-tracelessness condition has to be imposed on the fields in order to give rise to the correct number of spin- s degrees of freedom [30].

The difficulty in promoting these HS theories to interacting theories via coupling to gravity or electrodynamics is due to the fact that the presence of generic covariant derivatives in (2.2) violates the HS gauge symmetry. This in turn implies that the unphysical degrees of freedom are no longer eliminated and the theory becomes inconsistent. Despite of these negative results, Vasiliev has pioneered an approach towards a consistent coupling of HS fields to gravity, which is based on the introduction of an infinite-dimensional HS algebra [31]. The latter requires a frame-like formulation of HS fields, which mimics the vielbein formulation of general relativity rather than the metric-like formulation used in (2.2) [32]. More specifically, a spin-3 field, for instance, is described by e_μ^{ab} , being symmetric in the frame indices a, b , together with an analogue of the spin-connection $\omega_\mu^{ab,c}$. A closer inspection has, however, revealed that consistency of the HS algebra requires further fields, which are the so-called extra fields. These issues will be dealt with in later sections, but for the moment we just note that the resulting algebra will be derived from the enveloping algebra of the AdS_D algebra $\mathfrak{so}(D-1, 2)$, to which we turn in the next section.

2.2 A HIGHER-SPIN EXTENSION OF $\mathfrak{so}(D-1, 2)$

The starting point for the construction of an infinite-dimensional HS algebra is the AdS symmetry group $SO(D-1, 2)$. The Lie algebra of the latter is spanned by the anti-hermitian generators $M_{AB} = -M_{BA}$, $A = 0, 1, \dots, D-1, D+1$, obeying the commutation relations

$$\begin{aligned} [M_{AB}, M_{CD}] &= \eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC} + \eta_{AD}M_{BC} \\ &\equiv f_{AB,CD}{}^{EF}M_{EF} , \end{aligned} \quad (2.4)$$

where

$$\eta_{AB} = \text{diag}(-1, 1, 1, 1, 1, -1) , \quad f_{AB,CD}{}^{EF} = 4\delta_{[A}^E\eta_{B][C}\delta_{D]}^F] . \quad (2.5)$$

In the Lorentz basis, the $\mathfrak{so}(D-1, 2)$ commutation relations read

$$\begin{aligned} [M_{ab}, M_{cd}] &= \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac} + \eta_{ad}M_{bc} , \\ [M_{ab}, P_c] &= -2\eta_{c[a}P_{b]} , \quad [P_a, P_b] = M_{ab} . \end{aligned} \quad (2.6)$$

To define a Lie algebraic HS extension of $\mathfrak{so}(D-1, 2)$ it is convenient [33, 34] to introduce a set of bosonic vector oscillators y_A^i , where $i = 1, 2$ is an $\mathfrak{sp}(2)$ doublet index, obeying an associative non-commutative star product

$$y_A^i y_B^j = y_A^i y_B^j + \epsilon^{ij} \eta_{AB} , \quad [y_A^i, y_B^j]_* = 2\epsilon^{ij} \eta_{AB} , \quad (2.7)$$

where $\epsilon^{ij} = -\epsilon^{ji}$ denotes the invariant $\mathfrak{sp}(2)$ tensor, and we have introduced the bracket $[U, V]_* = U*V - V*U$. The star product of general functions $f(y)$ and $g(y)$ can be defined by the Moyal-Weyl formula

$$f(y) * g(y) = \exp \left(\frac{\partial}{\partial y_A^i} \frac{\partial}{\partial z_B^j} \epsilon^{ij} \eta_{AB} \right) f(y) g(z) \Big|_{z=y}, \quad (2.8)$$

which reduces for linear functions to (2.7).

Given the oscillators we can construct the generators of the commuting (Howe dual) algebras $\mathfrak{so}(D-1, 2)$ and $\mathfrak{sp}(2)$ as the bilinears

$$M_{AB} = \frac{1}{2} y_A^i y_{iB}, \quad K_{ij} = \frac{1}{2} y_i^A y_{jA}, \quad (2.9)$$

from which it indeed follows that $[K_{ij}, M_{AB}]_* = 0$. The construction of the HS Lie algebra $\mathfrak{ho}(D-1, 2)$ is based on the enveloping algebra of $\mathfrak{so}(D-1, 2)$. It is defined in terms of the oscillator as [33, 34]

$$\mathfrak{ho}(D-1, 2) = \left\{ T(y) : T^\dagger = -T, [K_{ij}, T]_* = 0 \right\}, \quad (2.10)$$

where $T(y)$ are arbitrary polynomials in the oscillator y_A^i , and the last condition singles out the $\mathfrak{sp}(2)$ singlets. This algebra is sometimes (perhaps misleadingly) referred to as the ‘off-shell HS algebra’ since it is generated by *trace-full generators*. On the other hand, starting from this algebra one may construct the corresponding ‘on-shell algebra’ where the generators are made traceless by factoring out the ideal in $\mathfrak{ho}(D-1, 2)$ spanned by elements of the form $K_{ij} * X^{ij}$ [33, 34, 11]. In the formulation of HS theories utilizing the approach of unfolded dynamics [15, 35], where an action principle is not needed, the on-shell algebra has been mostly used. It is only recently [34] that the importance of the off-shell algebra has been emphasized. In contrast, in an ordinary action formulation of HS theories, as the one in this paper, we believe that an algebra with trace-full generators is crucial. In the remaining of the paper we will avoid the term ‘off-shell algebra’.

The polynomial $T(y)$ appearing in (2.10) admits a level decomposition into monomials $T_\ell(y)$ (we associate the generators T_ℓ at level ℓ with spins $s = \ell + 2 = 2, 3, 4, \dots$)

$$T(y) = \sum_{\ell=0}^{\infty} T_\ell(y), \quad T_\ell(\lambda y) = \lambda^{2\ell+2} T_\ell(y). \quad (2.11)$$

The definition in terms of vector oscillators implies in particular that the algebra does not contain the full enveloping algebra spanned by polynomials in M_{AB} . Elements which vanish identically in the vector oscillator formulation belong to a certain ideal $\mathcal{I} \subset \mathcal{U}[\mathfrak{so}(D-1, 2)]$. For instance, the antisymmetric part $M_{[AB} M_{C]D}$ vanishes due to the $\mathfrak{sp}(2)$ identity $\epsilon^{[ij} \epsilon^{kl]} = 0$. This in turn implies that the generators of the HS algebra (2.10) are in specific Young tableaux. In other words, the $T_\ell(y)$ have an expansion in terms of $GL(D+1)$ tensors [34]

$$T_{A(s-1), B(s-1)} \equiv \mathbb{P}_{(s-1, s-1)} \left(M_{A_1 B_1} \cdots M_{A_{s-1} B_{s-1}} \right) \quad (2.12)$$

$$= \frac{1}{2^{2s-2}} \mathbb{P}_{(s-1, s-1)} \left(y_{A_1}^{i_1} y_{i_1 B_1} \cdots y_{A_{s-1}}^{i_{s-1}} y_{i_{s-1} B_{s-1}} \right), \quad (2.13)$$

where the $\mathfrak{so}(D-1, 2)$ generators appear at level 0.¹ We have introduced a notation in which $T_{A(n), B(n)} \equiv T_{A_1 \dots A_n, B_1 \dots B_n}$, each set of indices being totally symmetrized. $\mathbb{P}_{(s-1, s-1)}$ is a Young projector which imposes the symmetry of the two-row $GL(D+1)$ Young tableau (see appendix A for details)

$$\underbrace{\begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline \end{array}} \dots \underbrace{\begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline \end{array}}_{s-1} . \quad (2.14)$$

Later on we will need the generator in a $GL(D)$ ‘Lorentz’ basis. Splitting (2.14) accordingly for spin s , we find s generators, which schematically are in the tableaux

$$\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline \end{array}, \quad \dots, \quad \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline \end{array}. \quad (2.15)$$

More specifically, the generator $T_{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}}$ split into the series of generator $T_{a_1 \dots a_{s-1}}$ and $T_{a_1 \dots a_{s-1}, b_1 \dots b_t}$ for $1 \leq t \leq s-1$. The gauge fields $e_\mu^{a_1 \dots a_{s-1}}$ corresponding to the first generator will later be identified with the physical spin- s field, while the fields for the remaining $s-1$ generators are in the literature referred to as the auxiliary ($t=1$) and extra fields ($t>1$). However, for us this distinction between auxiliary and extra fields will be redundant and we will therefore henceforth refer to all fields with $t>0$ as auxiliary.

The complete set of commutation relations of the $\mathfrak{ho}(D-1, 2)$ algebra is not known in a closed form. Luckily, for the linearized spin- s analysis, to be treated in sec. 4 for an expansion around a spin-2 solution, it is sufficient to specify the spin-2 – spin- s commutation relations, which are entirely fixed by the representations theory of the AdS subalgebra $\mathfrak{so}(D-1, 2)$

$$[M_{AB}, T_{C(s-1), D(s-1)}]_* = -4(s-1) \mathbb{P}_{(s-1, s-1)} (\eta_{\underline{AC_{s-1}}} T_{BC(s-2), D(s-1)}) . \quad (2.16)$$

Let us finally mention that when commuting a spin- s generator with a spin- s' generator we obtain a sequence of generators with spins

$$s + s' - 2, s + s' - 4, \dots, |s - s'| + 2 . \quad (2.17)$$

Notice that only the $s=2$ subsector is closed.

3 CHERN-SIMONS THEORIES IN ODD DIMENSIONS

In this section we will introduce the formulation of Chern-Simons theories [24, 25] in general odd dimensions (for a review see [42]). The theory is specified once we give the algebra and the relevant invariant tensor. We will specify to AdS Lovelock gravity, with a focus on $D=5$, though the results directly extend to all odd dimensions (for the explicit formulas see [25].) Although there is no non-trivial propagation around the vacuum solution AdS_5 , interestingly, the theory also admits a simple AdS_4 solution [25] around which the graviton propagates. In sec. 4 we will analyze the linearized HS dynamics around this solution. Finally, in this section we will propose an invariant tensor for the full HS algebra.

¹There exists a further restriction of $\mathfrak{ho}(D-1, 2)$ to a minimal algebra containing only *even* spins $s=2, 4, 6, \dots$ [33, 34].

3.1 LOVELOCK GRAVITY AS $\mathfrak{so}(D-1, 2)$ GAUGE THEORY

In any odd dimension $D = 2n - 1$ a gauge-invariant Chern-Simons action can be defined, which is based on the invariant $2n$ -form $\langle F^n \rangle$ constructed out of the field strength F , with $\langle \rangle$ denoting an invariant symmetric tensor of degree n . More specifically, this expression is a total derivative and thus gives rise to a dynamically non-trivial theory only on the boundary, i.e. in one dimension less. The action can be written in closed form as

$$S_{\text{CS}} = \int_{M_{2n}} \langle F^n \rangle = \int_{M_{2n-1}} n \int_0^1 dt \langle A(tdA + t^2 A^2)^{n-1} \rangle, \quad (3.1)$$

where $M_{2n-1} = \partial M_{2n}$, and we left the wedge products implicit. The resulting Chern-Simons form in $D = 2n - 1$ is by construction gauge-invariant up to total derivatives. Explicitly, one has under arbitrary variations

$$\delta S_{\text{CS}} = n \int_{M_{2n-1}} \langle \delta A \wedge F^{n-1} \rangle, \quad (3.2)$$

i.e. gauge invariance under $\delta_\epsilon A_\mu = D_\mu \epsilon$ follows by the Bianchi identity.

For definiteness we focus on $D = 5$. The gauge field A then takes values in the Lie algebra of the group $SO(4, 2)$. Specifically, we write in an $SO(4, 1)$ covariant manner $A_\mu = e_\mu^a P_a + \frac{1}{2} \omega_\mu^{ab} M_{ab}$ in the basis above and define the invariant tensor to be

$$\langle M_{AB} M_{CD} M_{EF} \rangle = \varepsilon_{ABCDEF}. \quad (3.3)$$

Note that, as required, this tensor is *symmetric* in the sense that it stays invariant under exchange of M_{AB} with M_{CD} , etc. The $SO(4, 1)$ covariant field strength tensors in $F_{\mu\nu} = \frac{1}{2} \mathcal{R}_{\mu\nu}^{ab} M_{ab} + T_{\mu\nu}^a P_a$ read

$$\begin{aligned} \mathcal{R}_{\mu\nu}^{ab} &= R_{\mu\nu}^{ab} + \Lambda(e_\mu^a e_\nu^b - e_\nu^a e_\mu^b), \\ T_{\mu\nu}^a &= \partial_\mu e_\nu^a - \partial_\nu e_\mu^a + \omega_\mu^{ab} e_{\nu b} - \omega_\nu^{ab} e_{\mu b}, \end{aligned} \quad (3.4)$$

containing the Riemann tensor

$$R_{\mu\nu}^{ab} = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + \omega_\mu^a{}_c \omega_\nu^{cb} - \omega_\nu^a{}_c \omega_\mu^{cb}, \quad (3.5)$$

and the torsion tensor. The resulting Chern-Simons action can be written as [25]

$$\begin{aligned} S &= 3 \int_{M_5} \varepsilon_{a_1 \dots a_5} (e^{a_1} \wedge R^{a_2 a_3} \wedge R^{a_4 a_5} + \frac{2}{3} e^{a_1} \wedge e^{a_2} \wedge e^{a_3} \wedge R^{a_4 a_5} \\ &\quad + \frac{1}{5} e^{a_1} \wedge e^{a_2} \wedge e^{a_3} \wedge e^{a_4} \wedge e^{a_5}). \end{aligned} \quad (3.6)$$

We see that the action is the Einstein-Hilbert action with a cosmological constant (which we have set $\Lambda = 1$), extended by a $D = 5$ Lovelock term. To be more precise, it describes a theory with dynamical torsion. However, it is still consistent with the field equations to impose vanishing torsion in order to express the spin connection in terms of the vielbein. This in turn reduces the dynamical degrees of freedom to those of the metric, for which the Einstein equations read

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - 3\Lambda g_{\mu\nu} = \frac{1}{32\Lambda} (R R)_{\mu\nu}, \quad (3.7)$$

where we have introduced the abbreviation $(R R)_{\mu\nu} = \varepsilon_\mu^{\rho\sigma\lambda\delta} \varepsilon_{\nu\kappa\tau\gamma\phi} R_{\rho\sigma}^{\kappa\tau} R_{\lambda\delta}^{\gamma\phi}$.

As it stands, (3.6) seems to be a purely conventional type of Lovelock gravity, which is usually assumed to propagate the same number of degrees of freedom as Einstein gravity (five in $D = 5$). However, in this case the topological origin (3.1) actually gives rise to a somewhat unconventional behavior: Expanding (3.6) around the AdS_5 solution one infers that the quadratic term vanishes identically. In other words, a propagator around AdS_5 does not exist. This can be most easily understood from the general form of the equations of motion for the Chern-Simons action (3.1), which can be read off from (3.2)

$$g_{ABC} F^B \wedge F^C = 0, \quad (3.8)$$

where g_{ABC} denotes the invariant tensor and $\mathcal{A}, \mathcal{B}, \dots$, are the adjoint indices for a generic gauge group, which will be later on specified to $\mathfrak{ho}(4, 2)$. Since for an expansion around AdS_5 the curvature tensor in (3.4) vanishes in the background, there are no linear terms in (3.8) and thus no quadratic terms in the action. However, this should not be interpreted in the sense that the theory is devoid of local dynamics altogether, as is sometimes assumed of ‘topological’ actions like (3.1) in the literature. Indeed, the propagator around generic backgrounds does not vanish. Moreover, a careful Hamiltonian analysis of the dynamical content in [26, 27] has shown that, apart from degenerate sectors (like the maximally symmetric AdS_5 background), the theory consistently propagates a number of degrees of freedom depending on the dimension of the gauge group. In particular, the Lovelock-type gravity theory above has the expected five degrees of freedom.² Let us also stress that the degenerate sectors are only a measure-zero subspace within phase space [26, 27], and that even around such degenerate backgrounds some degrees of freedom can propagate, albeit fewer. One example has been given already in [25]: It is effectively an AdS_4 solution and reads $(\alpha, \beta = 0, 1, 2, 3)$

$$\bar{e}_\mu^a = \delta_\mu^a \frac{1}{1 - \frac{1}{4} \Lambda x^\alpha x_\alpha}, \quad \bar{\omega}_\mu^{\alpha\beta} = -\frac{\Lambda}{2} \frac{\delta_\mu^\alpha x^\beta - \delta_\mu^\beta x^\alpha}{1 - \frac{1}{4} \Lambda x^\alpha x_\alpha}, \quad \bar{e}_4^4 = \text{const.}, \quad (3.9)$$

which has vanishing torsion, $\bar{T}^a = 0$, and satisfies

$$\bar{R}^{\alpha\beta} + \Lambda \bar{e}^\alpha \wedge \bar{e}^\beta = 0, \quad \bar{R}^{\alpha 4} + \Lambda \bar{e}^\alpha \wedge \bar{e}^4 \neq 0. \quad (3.10)$$

By expanding around this solution, it has been shown that it propagates in particular a four-dimensional graviton [25].

3.2 INVARIANT TENSOR OF THE HIGHER-SPIN ALGEBRA

In order to construct the Chern-Simons action (3.1) based on $\mathfrak{ho}(D-1, 2)$, which extends standard Chern-Simons gravity, we have to find a completely symmetric tensor of degree $D-2$, which is invariant under the adjoint action of the HS algebra $\mathfrak{ho}(D-1, 2)$, and which reduces to the standard invariant (3.3) for the AdS -subalgebra $\mathfrak{so}(D-1, 2)$. Below we will propose a formula for the invariant tensor. However, while the vector oscillator formulation described in sec. 2.2 was required in order to establish existence and consistency of the HS algebra, it turns out not to be sufficient for the definition of a symmetric

²To be more precise, this counting applies only in case of vanishing torsion. Otherwise there are additional degrees of freedom [27].

invariants to $\mathfrak{ho}(D-1, 2)$. Instead, we will introduce a new star product, known as the BCH (Baker-Campbell-Hausdorff) star product or the Gutt star product [37, 38, 39].

Let us first briefly comment on the reasons why the formulation in terms of vector oscillators is incapable of reproducing the symmetric tensor (3.3). This is simply due to the fact that the oscillators automatically eliminate the totally antisymmetric part in the star product, $M_{[AB} * M_{CD} * M_{EF]} = 0$, since it involves an antisymmetrization over more than two $\mathfrak{sp}(2)$ indices. On the other hand, this exclusion guaranteed the appearance of generators entirely being in definite $(s-1, s-1)$ Young tableaux, or in other words, eliminated the ideals spanned by generators not in these Young tableaux. Here in contrast, by requiring an invariant tensor generalizing (3.3), we are, roughly speaking, assigning a non-zero value to certain parts in the ideal \mathcal{I} . Put differently, instead of using the invariance of the ideal, $[\mathfrak{ho}(D-1, 2), \mathcal{I}] \subset \mathcal{I}$, to set it to zero, we set it to constants, reducing in particular to (3.3).

To start with, we have to define a non-commutative star product directly in terms of the M_{AB} (here viewed as commuting coordinates), whose star commutator then yields the required $\mathfrak{so}(D-1, 2)$ algebra. This is the BCH star product, which is given by

$$F(M) \star G(M) = \exp \left(M_{AB} \Lambda^{AB} (\partial_N, \partial_{N'}) \right) F(N) G(N') \Big|_{N=M, N'=M}, \quad (3.11)$$

where ∂_N is a short-hand notation for $\partial/\partial N_{AB}$ and where $\Lambda^{AB} = -\Lambda^{BA}$ is defined through the relation

$$\exp Q \exp Q' = \exp \left(Q + Q' + \Lambda^{AB}(Q, Q') M_{AB} \right), \quad (3.12)$$

with $Q = Q^{AB} M_{AB}$ and $Q' = Q'^{AB} M_{AB}$ for some anti-symmetric tensors Q^{AB} and Q'^{AB} . It defines an associative product on the enveloping algebra [37]. By using the BCH formula

$$\exp Q \exp Q' = \exp \left(Q + Q' + \frac{1}{2} [Q, Q'] + \frac{1}{12} ([Q, [Q, Q']] + [Q', [Q', Q]]) + \dots \right), \quad (3.13)$$

we find the first few terms in the expansion to be

$$\begin{aligned} \Lambda^{AB} &= \frac{1}{2} f_{CD, EF}^{AB} Q^{CD} Q'^{EF} - \frac{1}{12} f_{CD, EF}^{GH} f_{GH, IJ}^{AB} Q^{CD} Q'^{EF} (Q^{IJ} - Q'^{IJ}) + \dots \\ &= 2(QQ')^{[AB]} - \frac{2}{3} [Q, Q']^{[A|C|} (Q_C^{B]} - Q'_C^{B]}) + \dots, \end{aligned} \quad (3.14)$$

where $(QQ')^{AB} = Q^{AC} Q'_C{}^B$, $[Q, Q']^{AB} = (QQ')^{AB} - (Q'Q)^{AB}$ and where $f_{AB, CD}{}^{EF}$ are the structure constants defined in (2.5). The first terms in the product (3.11) consequently becomes

$$\begin{aligned} F(M) \star G(M) &= F(M) G(M) + 2M_{AB} \partial^{AC} F \partial_C{}^B G \\ &\quad + \frac{2}{3} M_{AB} \left(\partial^{AC} \partial^{BD} F \partial_{CD} G - (\partial^2)^{AC} F \partial^B{}_C G \right. \\ &\quad \left. + \partial^{AC} \partial^{BD} G \partial_{CD} F - (\partial^2)^{AC} G \partial^B{}_C F \right) + \dots. \end{aligned} \quad (3.15)$$

The definition of the HS generators in (2.12) extends immediately. However, whereas the realization in terms of the vector oscillator automatically imposes the Young tableau

symmetries $(s-1, s-1)$, here we need to Young project explicitly.³ Hence, all elements of the enveloping algebra which belong to other Young tableaux are modded out.

The star-products between a spin-2 generator and spin- s generator $T_{C(s-1), D(s-1)}$ read

$$M_{AB} \star M_{CD} = M_{AB}M_{CD} - 2\eta_{C[A}M_{B]D} , \quad (3.16)$$

$$\begin{aligned} M_{AB} \star T_{C(s-1), D(s-1)} &= M_{AB}T_{C(s-1), D(s-1)} \\ &\quad - 2(s-1)\mathbb{P}_{(s-1, s-1)}(\eta_{AC_{s-1}}T_{BC(s-2), D(s-1)}) \\ &\quad + \text{double contractions} , \end{aligned} \quad (3.17)$$

where $\mathbb{P}_{(s-1, s-1)}$ is a Young projector. The commutation relations in (2.4) and (2.16) follow readily by defining the bracket $[U, V]_\star = U \star V - V \star U$, since we know that $[M_{AB}, F(M)]_\star = 4M_{C[A}\partial^C_{B]}F(M)$; see Eq. (B.2) in appendix B.

Let us now proceed with defining the symmetric invariant tensors of the HS algebra. Given an element $F(M)$ of the enveloping algebra $\mathcal{U}[\mathfrak{so}(D-1, 2)]$, we define the operation ‘tr’ given by evaluation at $M_{AB} = 0$

$$\text{tr}(F(M)) := F(0) . \quad (3.18)$$

However, although the analogue of this operation for the vector oscillator described in sec. 2.2 constitutes a proper (super) trace [40, 6, 41], it is easy to realize that the bilinear $\text{tr}(F(M) \star G(M))$ vanishes identically in our case (see also the comments in footnote 4 below). To obtain a sensible non-zero trace, we need to insert $GL(D+1)$ -invariant differential operators into the trace (3.18) *cf.* the results in Ref. [36]. A natural $GL(D+1)$ -invariant differential operator is constructed out of $n = (D+1)/2$ derivatives contracted with the totally anti-symmetric tensor $\epsilon_{A_1 \dots A_{D+1}}$. We propose the following sequence of traces ‘ Tr_k ’, for $k = 1, 2, 3, \dots$

$$\text{Tr}_k(F(M)) = \text{tr}(\Delta^k[F(M)]) , \quad (3.19)$$

$$\Delta = \epsilon_{A_1 \dots A_n B_1 \dots B_n} \frac{\partial}{\partial M_{A_1 B_1}} \cdots \frac{\partial}{\partial M_{A_n B_n}} , \quad (3.20)$$

with ‘tr’ as in (3.18).⁴ These traces are cyclic

$$\text{Tr}_k(F \star G) = \text{Tr}_k(G \star F) , \quad (3.21)$$

for generic elements $F(M)$ and $G(M)$ of the enveloping algebra, which will be proven in appendix B.

We now define the symmetric trilinear for three generators (2.12) of the HS algebra $\mathfrak{ho}(4, 2)$ of spins s, s' and s'' to be

$$\langle T_s, T_{s'}, T_{s''} \rangle := \sum_{k=1}^{\infty} \alpha_k \text{Tr}_k(\{T_s, T_{s'}\}_\star \star T_{s''}) , \quad (3.22)$$

³Note that under the projector $\mathbb{P}_{(s-1, s-1)}$ the use of the star product or the point-wise (‘classical’) product is immaterial. For instance, for spin 3 we have $T_{AB, CD} = \mathbb{P}_{(2, 2)}(M_{AC} \star M_{BD}) = \mathbb{P}_{(2, 2)}(M_{AC}M_{BD})$.

⁴The oscillator algebra based on (2.8) admits a natural graded (super) trace $\text{tr}_y f(y) = f(0)$, such that $\text{tr}_y(f(y) \star g(y)) = \text{tr}_y(g(-y) \star f(y))$ [40, 6, 41]. Using this trace we can construct the anti-symmetric invariants of the $\mathfrak{ho}(D-1, 2)$ algebra. However, for the reasons explained above, even ‘dressing’ this trace with derivative operators analogous to (3.20), cannot give rise to a non-vanishing symmetric combination.

where $\{T_s, T_{s'}\}_\star = T_s \star T_{s'} + T_{s'} \star T_s$ and α_k are arbitrary coefficients. This definition generalizes directly to an n -form for $D = 2n - 1$. The total symmetry of (3.22) follows from (3.21) and the associativity of the BCH star product.

At this stage we have to note that, strictly speaking, the cyclicity (3.21) is not sufficient to prove invariance of (3.22), since the commutator with respect to the BCH star product potentially contains ideal terms. However, for the linearization in case of a spin-3 field to be analyzed below, one can check explicitly that the tensor is invariant to that order. So we expect (3.22) to be invariant under the adjoint action of the full $\mathfrak{ho}(4, 2)$, which, furthermore, might fix the free coefficients α_k .

The definition (3.22) will reproduce the symmetric spin-2 trilinear (3.3) provided we choose the first coefficient to be $\alpha_1 = 1/12$. Further, it follows that the spin-2, spin-2, spin- s invariant vanishes for $s > 2$ once the symmetries imposed by the Young projector of the spin- s generator are taken into account,

$$\langle M_{AB}, M_{CD}, T_{E(s-1), F(s-1)} \rangle = 0. \quad (3.23)$$

This relation guarantees that the equations of motion for the HS fields (see (3.8)) will not contain a term depending only on the space-time curvature, which in turn implies that the spin-2 field does not provide a source for the HS fields. Put differently, it is consistent with the field equations to set the HS fields to zero.

Only the first trace Tr_1 enters the linearized spin-3 analysis which we will focus on below. The relevant invariant in $D = 5$ is given by

$$\langle M_{AB}, T_{CD, EF}, T_{GH, IJ} \rangle = -2\mathbb{P}_{(2,2)}\mathbb{P}'_{(2,2)}(\epsilon_{ABCEGI} \eta_{DH} \eta_{FJ}), \quad (3.24)$$

where the projectors impose the symmetries of the two spin-3 generators. We note that up to an overall constant, the invariant (3.24) is the only possible term which is consistent with the imposed Young symmetries.

Up to now we established the existence of a HS Lie algebra and an associated symmetric invariant tensor. This in turn is sufficient to define a consistent HS Chern-Simons action, which in, say, $D = 5$ is given by

$$S = \int_{M_5} \left\langle \mathcal{W} \wedge d\mathcal{W} \wedge d\mathcal{W} + \frac{3}{2} d\mathcal{W} \wedge \mathcal{W} \wedge \mathcal{W} \wedge \mathcal{W} + \frac{3}{5} \mathcal{W} \wedge \mathcal{W} \wedge \mathcal{W} \wedge \mathcal{W} \wedge \mathcal{W} \right\rangle. \quad (3.25)$$

Here \mathcal{W} denotes the gauge field taking values in $\mathfrak{ho}(4, 2)$. It contains by construction the Lovelock gravity discussed in sec. 3.1, corresponding to the subalgebra $\mathfrak{so}(4, 2)$. Note that all the complexity of this theory is encoded in the infinite-dimensional Lie algebra $\mathfrak{ho}(4, 2)$ and the symmetric tensor. By virtue of the consistency of $\mathfrak{ho}(4, 2)$ and the existence of the tri-linear tensor, this action is by construction invariant under an exact HS symmetry at the full non-linear level, i.e. it satisfies requirement (i) in the introduction. However, due to the fact that the Lie brackets of $\mathfrak{ho}(4, 2)$ are not known explicitly, at this stage the action (3.25) cannot be rewritten in a closed form in terms of the physical HS fields. Fortunately a linearized analysis can be performed, and in the next section we will show that one recovers indeed the correct free field limit, thus proving that (3.25) satisfies also condition (ii).

4 DYNAMICAL ANALYSIS

In this section we will discuss some aspects of the dynamical content of the constructed HS theory. As it stands, the HS action (3.25) describes a theory with propagating gravitational torsion, so we expect also the HS torsions (which will be defined below) to propagate. Since the dynamics of these kind of theories is much less understood, we take here a pragmatic point of view, i.e. we impose vanishing torsion, which is compatible with the equations of motion though it is not enforced by them. For simplicity our focus will be on the first non-trivial case, *viz.* spin-3, which we believe exhibits generic features present for arbitrary spin.

4.1 LINEARIZATION AND CONSTRAINTS FOR SPIN-3

We first note that, as in the purely gravitational case, an expansion around AdS_5 does not give rise to a non-trivial propagator. This can be seen by inspecting the equations of motion (3.8). Up to first order they are of the form

$$g_{ABC}\mathcal{R}_{\text{AdS}}^B \wedge R_{\text{HS}}^C = 0, \quad (4.1)$$

where R_{HS} denotes the linearized HS contribution. As the AdS -covariant field strength vanishes in the AdS background, $\mathcal{R}_{\text{AdS}} = 0$, the equations are identically satisfied at the first order and do not lead to any perturbative dynamics.

Instead we will first keep the discussion generic and later focus on an expansion around the $AdS_4 \times S^1$ solution discussed in sec. 3.1. For this we have to know the HS algebra explicitly. Fortunately, for an expansion around a given background geometry, only the commutators between spin-2 and spin- s generator enter, while the mutual interactions between the different HS fields are not relevant. The spin-3 generator is given by $T_{AB,CD}$, corresponding to the Young tableau $\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}$, and it closes according to (2.16) with the spin-2 generator as⁵

$$\begin{aligned} [M_{AB}, T_{CD,EF}] &= -2 \left(\eta_{\underline{AC}} \underline{T_{BD,EF}} + \eta_{\underline{AD}} \underline{T_{CB,EF}} + \eta_{\underline{AE}} \underline{T_{CD,BF}} + \eta_{\underline{AF}} \underline{T_{CD,EB}} \right), \\ &= -8 \eta_{\underline{A[C} \underline{T_{B|D,EF}}} \rangle. \end{aligned} \quad (4.2)$$

Here curly brackets denote $(2, 2)$ Young projection, while in the following they also indicate symmetrization according to the Hook tableau, etc. (see appendix A). In a $GL(5)$ covariant basis, the spin-3 generators are given by $T_{ab} = T_{ab,66}$, $T_{ab,c} = T_{ab,c6}$ and $T_{ab,cd}$, and their algebra reads

$$\begin{aligned} [M_{ab}, T_{cd}] &= -4 \eta_{\underline{a[c} \underline{T_{b|d}}} \rangle, & [M_{ab}, T_{cd,e}] &= -4 \eta_{\underline{a[c} \underline{T_{b|d,e}}} \rangle + 4 \eta_{\underline{a[c} \underline{T_{de}}, b} \rangle, \\ [M_{ab}, T_{cd,ef}] &= -8 \eta_{\underline{a[c} \underline{T_{b|d,ef}}} \rangle, & [P_a, T_{bc}] &= 2 T_{bc,a}, \\ [P_a, T_{bc,d}] &= 3 \eta_{\underline{a[b} \underline{T_{cd}}} \rangle - T_{ad,bc}, & [P_a, T_{bc,de}] &= 8 \eta_{\underline{a[b} \underline{T_{cd,e}}} \rangle. \end{aligned} \quad (4.3)$$

Here we take the brackets $[T, T]$ to be vanishing, even though in the full HS algebra they close into spin-4 generator. However, in the linearization these spin-4 fields decouple, and, indeed, this truncation defines a consistent Lie algebra.

⁵In the sequel we will drop the \star subscript on the commutators.

Next we linearize the HS gauge field as⁶

$$\mathcal{W}_\mu = \bar{e}_\mu{}^a P_a + \frac{1}{2} \bar{\omega}_\mu{}^{ab} M_{ab} + \kappa \left(\frac{1}{2} e_\mu{}^{ab} T_{ab} + \frac{1}{3} \omega_\mu{}^{ab,c} T_{ab,c} + \frac{1}{12} \omega_\mu{}^{ab,cd} T_{ab,cd} \right), \quad (4.4)$$

where $\bar{e}_\mu{}^a$ and $\bar{\omega}_\mu{}^{ab}$ are vielbein and spin connection of the background geometry. Moreover, we consistently omitted contributions from all fields with spin $s > 3$. $e_\mu{}^{ab}$ will later be identified with the spin-3 field, while $\omega_\mu{}^{ab,c}$ and $\omega_\mu{}^{ab,cd}$ are auxiliary fields that have to be eliminated by means of constraints. It will turn out that these constraints are analogous to the torsion constraint of general relativity. As the torsion tensor appears as part of the field strength in (3.4), we will determine the required constraints in the HS case by computing the non-abelian field strength based on the algebra (4.3). We find

$$\begin{aligned} \mathcal{F}_{\mu\nu} &= \partial_\mu \mathcal{W}_\nu - \partial_\nu \mathcal{W}_\mu + [\mathcal{W}_\mu, \mathcal{W}_\nu] \\ &= \bar{T}_{\mu\nu}{}^a P_a + \frac{1}{2} \bar{\mathcal{R}}_{\mu\nu}{}^{ab} M_{ab} + \kappa \left(\frac{1}{2} T_{\mu\nu}{}^{ab} T_{ab} + \frac{1}{3} T_{\mu\nu}{}^{ab,c} T_{ab,c} + \frac{1}{12} R_{\mu\nu}{}^{ab,cd} T_{ab,cd} \right) + \mathcal{O}(\kappa^2). \end{aligned} \quad (4.5)$$

Here $\bar{T}_{\mu\nu}{}^a$ denotes the background torsion, which we assume to vanish, while $\bar{\mathcal{R}}_{\mu\nu}{}^{ab}$ is the *AdS*-covariant background curvature tensor. The linearized HS field strengths read

$$\begin{aligned} T_{\mu\nu}{}^{ab} &= \bar{D}_\mu e_\nu{}^{ab} - \bar{D}_\nu e_\mu{}^{ab} + \omega_\mu{}^{ab,c} \bar{e}_{\nu c} - \omega_\nu{}^{ab,c} \bar{e}_{\mu c}, \\ T_{\mu\nu}{}^{ab,c} &= \bar{D}_\mu \omega_\nu{}^{ab,c} - \bar{D}_\nu \omega_\mu{}^{ab,c} + \omega_\mu{}^{ab,cd} \bar{e}_{\nu d} - \omega_\nu{}^{ab,cd} \bar{e}_{\mu d} \\ &\quad + 3e_\mu{}^{ab} \bar{e}_\nu{}^{c} - 3e_\nu{}^{ab} \bar{e}_\mu{}^c, \\ R_{\mu\nu}{}^{ab,cd} &= \bar{D}_\mu \omega_\nu{}^{ab,cd} - \bar{D}_\nu \omega_\mu{}^{ab,cd} + 4\omega_\mu{}^{ab,c} \bar{e}_\nu{}^d - 4\omega_\nu{}^{ab,c} \bar{e}_\mu{}^d, \end{aligned} \quad (4.6)$$

where \bar{D}_μ denotes the background Lorentz covariant derivative, which read on the different fields

$$\begin{aligned} \bar{D}_\mu e_\nu{}^{ab} &= \partial_\mu e_\nu{}^{ab} + 2\bar{\omega}_\mu{}^{a}{}_c e_\nu{}^{c|b}, \\ \bar{D}_\mu \omega_\nu{}^{ab,c} &= \partial_\mu \omega_\nu{}^{ab,c} + 2\bar{\omega}_\mu{}^{a}{}_d \omega_\nu{}^{d|b,c} - 2\bar{\omega}_\mu{}^{a}{}_d \omega_\nu{}^{bc,d}, \\ \bar{D}_\mu \omega_\nu{}^{ab,cd} &= \partial_\mu \omega_\nu{}^{ab,cd} + 4\bar{\omega}_\mu{}^{a}{}_e \omega_\nu{}^{e|b,cd}. \end{aligned} \quad (4.7)$$

Before we turn to the constraints let us discuss the spin-3 symmetries, under which the field strengths above stay invariant. Under a non-abelian gauge transformation $\delta \mathcal{W}_\mu = D_\mu \epsilon = \partial_\mu \epsilon + [\mathcal{W}_\mu, \epsilon]$, with Lie algebra valued transformation parameter ϵ given in the spin-3 case by

$$\epsilon = \xi^a P_a + \frac{1}{2} \Lambda^{ab} M_{ab} + \kappa \left(\frac{1}{2} \epsilon^{ab} T_{ab} + \frac{1}{3} \epsilon^{ab,c} T_{ab,c} + \frac{1}{12} \epsilon^{ab,cd} T_{ab,cd} \right), \quad (4.8)$$

we find the following variations (ignoring background diffeomorphisms and Lorentz transformations)

$$\begin{aligned} \delta_\epsilon e_\mu{}^{ab} &= \bar{D}_\mu \epsilon^{ab} - \epsilon^{ab,c} \bar{e}_{\mu c}, \\ \delta_\epsilon \omega_\mu{}^{ab,c} &= \bar{D}_\mu \epsilon^{ab,c} - 3\epsilon^{ab} \bar{e}_\mu{}^c - \epsilon^{ab,cd} \bar{e}_{\mu d}, \\ \delta_\epsilon \omega_\mu{}^{ab,cd} &= \bar{D}_\mu \epsilon^{ab,cd} - 4\epsilon^{ab,c} \bar{e}_\mu{}^d. \end{aligned} \quad (4.9)$$

⁶The unit-strength normalizations follow from the Hook length formula [43].

Note that the gauge transformations with parameter $\epsilon^{ab,c}$ and $\epsilon^{ab,cd}$ corresponding to the auxiliary fields act as Stückelberg shift symmetries.

Next we are going to discuss the constraints. We will see that imposing the conditions⁷

$$T_{\mu\nu}{}^{ab} = 0, \quad T_{\mu\nu}{}^{ab,c} = 0, \quad (4.10)$$

allows to express $\omega_\mu{}^{ab,c}$ in terms of the physical spin-3 field $e_\mu{}^{ab}$ and its first derivative and $\omega_\mu{}^{ab,cd}$ in terms of $\omega_\mu{}^{ab,c}$ and its first derivatives. In turn, $\omega_\mu{}^{ab,cd}$ is a function of $e_\mu{}^{ab}$ and its first and second derivatives. The latter can be inserted into the third of the equations (4.6), which then yields the HS generalization of the Riemannian curvature tensor. Therefore the spin-3 curvature tensor will be of third order in the derivatives of the spin-3 field. This procedure can be generalized to arbitrary spin- s fields, whose curvature tensor will thus contain the s -th derivative of the physical spin- s field. (For trace-less tensors in $D = 4$ spinorial form this analysis has been done in [44], while a cohomological analysis in D dimensions can be found in [45, 11].) This corresponds to the hierarchy of deWit–Freedman connections found in the metric-like formulation [30]. Since the equations of motion will necessarily impose conditions on the HS Riemann tensor, this implies that the field equations are in the linearization already of higher derivative order. So at first sight we seem to have little chance to recover the required 2nd order Frønsdal equations. However, in flat space it has been shown that the Riemann tensor is a curl (‘Damour-Deser identity’ [49]) and that it can therefore be locally integrated, giving rise to the Frønsdal equations in the so-called compensator formulation [34, 50]. Here we will prove that this generalizes to AdS .

Let us now turn to the constraints. From the first of the equations (4.10) we conclude

$$\omega^{dbc,a} - \omega^{abc,d} = \Omega_1^{ad,bc}, \quad (4.11)$$

where the curved index on $\omega_\mu{}^{ab,c}$ has been converted into a flat index by means of the background vielbein, and we have introduced a HS generalization of the coefficients of anholonomy,

$$\Omega_1^{ab,cd} = \bar{e}^{\mu a} \bar{e}^{\nu b} (\bar{D}_\mu e_\nu{}^{cd} - \bar{D}_\nu e_\mu{}^{cd}). \quad (4.12)$$

By permuting the indices in (4.11), one finds the expression

$$\omega_\mu{}^{bc,d} = \frac{1}{2} \bar{e}_{\mu a} (\Omega_1^{a(b,c)d} - \Omega_1^{ad,bc} + \Omega_1^{d(b,c)a}) + \xi_\mu{}^{bc,d}, \quad (4.13)$$

where

$$\xi_\mu{}^{bc,d} = \frac{1}{4} \bar{e}_{\mu a} (\omega^{abc,d} + \omega^{bda,c} + \omega^{cda,b} + \omega^{dbc,a}). \quad (4.14)$$

To understand the significance of $\xi_\mu{}^{ab,c}$, we first note that a priori (4.13) lives in the Young tableaux

$$\square \otimes \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \square & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \square & & \\ \hline \end{array}. \quad (4.15)$$

⁷Note that the first constraint allows to identify the background diffeomorphisms with the gauge transformations generated by ξ^a in the sense that the latter read on $e_\mu{}^{ab}$, up to local Lorentz and Stückelberg transformations, $\delta_\xi e_\mu{}^{ab} = \mathcal{L}_\xi e_\mu{}^{ab}$, where \mathcal{L}_ξ denotes the Lie derivative with respect to the vector field $\xi^\mu = \bar{e}^\mu_a \xi^a$.

It follows from (4.14) that ξ is in the window tableau, i.e. $(1 - \mathbb{P}_{(2,2)})\xi = 0$. In the following we will have to treat ξ as an independent field. One can easily check that (4.13) solves (4.11) for arbitrary ξ , by using the window property of the latter. In fact, we will see that the inclusion of this auxiliary field is necessary in order for the composite connection $\omega_\mu^{bc,d}(e, \xi)$ to reproduce the correct transformation behavior in (4.9).

From now on we will specify the geometry to *AdS*, since this is the case we are interested in later on.⁸ Specifically, the background-covariant derivative \bar{D}_μ reduces to the *AdS*-covariant derivative ∇_μ , characterized by

$$[\nabla_\mu, \nabla_\nu]V_\rho = \frac{1}{L^2}(g_{\nu\rho}V_\mu - g_{\mu\rho}V_\nu) , \quad (4.16)$$

with the *AdS* metric $g_{\mu\nu}$ of radius L , which in our conventions is $L = 1$. Applying the first equation of (4.9) to (4.13), one can verify by use of (4.16) that $\omega_\mu^{ab,c}(e, \xi)$ transforms exactly as required by the second equation of (4.9), if one defines

$$\delta_\epsilon \xi_{\mu\ \nu\rho, \sigma} = \nabla_{\langle\mu} \epsilon_{\nu\rho, \sigma\rangle} - 3\epsilon_{\langle\nu\rho} g_{\sigma\mu\rangle} - \epsilon_{\nu\rho, \sigma\mu} . \quad (4.17)$$

In particular one sees that this transformation rule is consistent with the window symmetry of $\xi_\mu^{ab,c}$.

The second torsion constraint in (4.10) can now be solved in a similar fashion. For our purposes it will, however, be sufficient to perform this analysis in a gauged-fixed formulation (for *AdS* backgrounds). This will effectively reduce the field content to the completely symmetric spin-3 field, given in a metric-like formulation by

$$h_{\mu\nu\rho} := \bar{e}_{(\mu}^a \bar{e}_{\nu}^b e_{\rho)ab} . \quad (4.18)$$

Specifically we use the Stückelberg shift symmetry in (4.9) parametrized by $\epsilon^{ab,c}$ to gauge the hooked part \boxplus of $e_{\mu ab}$ to zero (see (A.3) in app. A). However, this gauge fixing will be violated by a generic spin-3 transformation, and so one has to add a compensating shift transformation with parameter $\epsilon_{ab,c} = \nabla_{\langle c} \epsilon_{ab\rangle}$. Under this residual gauge symmetry only the completely symmetric part of $e_{\mu\nu\rho}$ transforms, namely as

$$\delta_\epsilon h_{\mu\nu\rho} = \nabla_{(\mu} \epsilon_{\nu\rho)} , \quad (4.19)$$

as required in the free limit (see sec. 2.1). Furthermore, from (4.17) we infer, that also $\xi_\mu^{ab,c}$ is subject to a Stückelberg shift symmetry with transformation parameter in \boxplus . Therefore it can be gauged away completely, which in turn requires a compensating transformation with

$$\epsilon_{\nu\rho, \mu\sigma} = \nabla_{\langle\mu} \nabla_{\sigma} \epsilon_{\nu\rho\rangle} - 3\epsilon_{\langle\nu\rho} g_{\sigma\mu\rangle} . \quad (4.20)$$

In total, after gauge-fixing the spin-3 connections will depend only on the completely symmetric part of $e_{\mu ab}$.

To solve the second torsion constraint we derive from (4.6) for $\omega_\nu^{ab,cd}$ in flat indices:

$$\omega^{a\ bc,de} - \omega^{e\ bc,da} = \Omega_2^{ea\ bc,d} , \quad (4.21)$$

⁸Note, however, that the analysis performed here holds in an arbitrary dimension, i.e. it applies in particular to *AdS*₅ as well as the *AdS*₄ geometry we will discuss below.

where

$$\Omega_2^{ab\,cd,e} = \bar{e}^{a\mu}\bar{e}^{b\nu}(\bar{D}_\mu\omega_\nu^{cd,e} + 3e_\mu^{\langle cd}\bar{e}_\nu^{e\rangle} - (\mu \leftrightarrow \nu)) . \quad (4.22)$$

We find the solution

$$\omega_\mu^{ab,cd} = -\frac{1}{2}\bar{e}_{\mu f}(\Omega_2^{fa\,cd,b} + \Omega_2^{fb\,cd,a} + \Omega_2^{fc\,ab,d} + \Omega_2^{fd\,ab,c}) . \quad (4.23)$$

To verify that this is a solution it is not sufficient to use the symmetries of the $\Omega_2^{ab\,cd,e}$, but instead the explicit expression given by (4.22) and (4.13) together with the *AdS* relation (4.16) is required.

4.2 SPIN-3 FIELD EQUATIONS

Let us now turn to the equations of motion. We specify to the *AdS*₄ background discussed in sec. 3.1. Moreover, we set all components of e_μ^{ab} which have a leg in the fifth dimension to zero. In other words, we are not considering the dynamics of Kaluza-Klein scalars and vectors, etc., in order to simplify the analysis. Though in the full non-linear theory this would most likely not be a consistent Kaluza-Klein truncation, in the linearization this is justified since the different fields decouple.

Using the explicit form of the invariant tensor in (3.24) we see that, after imposing the constraint, the only non-trivial part of the equations of motion (4.1) requires the free index to take values in the hook tableau. Moreover, we have seen in eq. (3.23) that setting the background spin-3 field to zero is consistent with its equations of motion, which we implicitly assumed already in the expansion (4.4).

Specifically, by use of (3.24) we have

$$0 = \mathbb{P}_{\square\square}(\epsilon_{abcde}\bar{R}^{ab} \wedge R_{f,\,e}^d{}_h) = \frac{1}{2}(\epsilon_{abcde}\bar{R}^{ab} \wedge R_{f,\,h}^d{}_e + \epsilon_{abhde}\bar{R}^{ab} \wedge R_{f,\,c}^d{}_e) , \quad (4.24)$$

where we used in the second equation the projector in (ch, f) . Specifying now to flat *AdS*₄ indices $a = (\alpha, 4)$,⁹ and using (3.10) this implies

$$\epsilon_{\alpha\beta\gamma\delta}\bar{e}^\alpha \wedge R_{\epsilon,\,\zeta}^\gamma{}^\delta + (\zeta \leftrightarrow \beta) = 0 . \quad (4.25)$$

This yields in components by use of the identity $\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}\bar{e}_\mu^\alpha = 3!e\bar{e}_{[\beta}^\nu\bar{e}_\gamma^\rho\bar{e}_{\delta]}^\sigma$ and after relabelling the indices

$$R_{\mu\lambda}{}^\lambda{}_{\nu,\rho\sigma} - R_{\mu\lambda}{}^\lambda{}_{\sigma,\rho\nu} + R_{\lambda\delta}{}^\lambda{}_{\nu,\,\delta}{}^\sigma g_{\mu\rho} + (\mu \leftrightarrow \sigma) = 0 . \quad (4.26)$$

Taking the μ, ρ trace implies that the double trace of the Riemann tensor vanishes. We prove in appendix C that the final equation is equivalent to the condition that any single trace of the Riemann tensor, i.e. the spin-3 analogue of the Ricci tensor, vanishes. It turns out that a convenient choice is the following:

$$R_{\mu\nu\,\alpha\beta,\,\delta}{}^\delta = 0 . \quad (4.27)$$

⁹Indices μ, ν, \dots , denote *D*-dimensional space-time indices. We hope it will not source any confusion that we specify them in this section to curved *AdS*₄ indices.

Next we are going to analyze this equation in more detail. By inserting (4.13) into (4.23) and using (4.6) one finds the explicit expressions (in curved indices)

$$\nabla_\mu K_{\nu\rho\sigma} + g_{\sigma\nu}(\nabla_\rho h'_\mu - \nabla \cdot h_{\mu\rho}) + g_{\rho\nu}(\nabla_\sigma h'_\mu - \nabla \cdot h_{\mu\sigma}) - (\mu \leftrightarrow \nu) = 0, \quad (4.28)$$

where we defined

$$K_{\nu\rho\sigma} = \square h_{\nu\rho\sigma} - \nabla_\rho \nabla \cdot h_{\sigma\nu} - \nabla_\sigma \nabla \cdot h_{\rho\nu} + \nabla_{(\rho} \nabla_{\sigma)} h'_\nu - (D-3)h_{\nu\rho\sigma} - g_{\nu(\rho} h'_{\sigma)} - 3g_{\rho\sigma} h'_\nu. \quad (4.29)$$

Here we left the space-time dimension generic, though in our case it is $D = 4$. As outlined above, we are going to show that these 3rd order differential equations can locally be integrated to give effectively rise to sensible 2nd order field equations. We first note that, in contrast to Minkowski space, on AdS a vanishing curl cannot locally be integrated to a gradient, since the covariant derivatives do not commute. However, for $K_{\nu\rho\sigma}$ symmetric in ρ, σ a condition like

$$\nabla_{[\mu} K_{\nu]\rho\sigma} = -2g_{\rho[\mu} \beta_{\nu]\sigma} - 2g_{\sigma[\mu} \beta_{\nu]\rho}, \quad (4.30)$$

with symmetric $\beta_{\mu\nu}$, can be solved by $K_{\mu\nu\rho} = \nabla_\mu \beta_{\nu\rho}$, as follows after reinsertion from (4.16). Comparing with (4.28) we see that the equations of motion have almost this form, except that the $\beta_{\mu\nu}$ derived like this are not symmetric. If the latter are symmetrized by hand, additional terms have to be added to the ansatz for $K_{\mu\rho\sigma}$, which in turns implies that it is no longer a pure gradient. Moreover, an integration constant has to be carefully taken into account. Altogether one finds that

$$K_{\nu\rho\sigma} = \nabla_\nu \beta_{\rho\sigma} + g_{\nu\rho}(h'_\sigma - \nabla_\sigma \alpha) + g_{\nu\sigma}(h'_\rho - \nabla_\rho \alpha) \quad (4.31)$$

solves (4.28), where

$$\beta_{\mu\nu} = \nabla \cdot h_{\mu\nu} - \nabla_\mu h'_\nu - \nabla_\nu h'_\mu + \nabla_\mu \nabla_\nu \alpha - 2g_{\mu\nu} \alpha, \quad (4.32)$$

and α is the integration constant. Then (4.31) can be rewritten by use of the explicit expression in (4.29) as

$$\mathcal{F}_{\mu\nu\rho}^{\text{AdS}} = \nabla_{(\mu} \nabla_{\nu} \nabla_{\rho)} \alpha - 4g_{(\mu\nu} \nabla_{\rho)} \alpha, \quad (4.33)$$

where \mathcal{F}^{AdS} is the AdS Frønsdal operator defined in sec. 2.1. To understand the significance of α we note that the equation (4.28) is by construction spin-3 invariant. However, by locally integrating, this invariance would be lost, if not a non-trivial transformation behavior is assigned to the ‘integration constant’ α . In fact, (4.33) is only invariant if

$$\delta_\epsilon \alpha = \epsilon'. \quad (4.34)$$

This shift symmetry can now be used to set $\alpha = 0$, such that (4.33) reduces to the Frønsdal equation on AdS , the latter being invariant under all trace-less spin-3 transformations. Thus we correctly recovered the required free spin-3 equations. The formulation (4.33) with its invariance under *trace-full* transformations and the appearance of the so-called compensator α in fact coincides completely with the construction of Francia and Sagnotti [46, 47, 48].

5 CONCLUSIONS AND OUTLOOK

By virtue of the Yang-Mills gauge invariance of Chern-Simons actions in any odd dimension, the HS theories constructed in this paper provide a consistent coupling to gravity in the sense that the free HS symmetry $\delta_\epsilon h_{\mu_1 \dots \mu_s} = \nabla_{(\mu_1} \epsilon_{\mu_2 \dots \mu_s)}$ gets deformed to an exact symmetry of the full non-linear theory. In other words, condition (i) raised in the introduction is satisfied by construction. Moreover, contrary to what is sometimes implicitly assumed, these ‘topological’ actions do possess propagating degrees of freedom for $D > 3$. By linearizing around the AdS_4 solution found in [25], we verified explicitly that this is the case especially in the presence of HS fields. We recovered the correct free field equations in the first non-trivial case of a spin-3 field. For this we showed that on the subsector of vanishing spin-3 torsion the field equations, though being 3rd order differential equations, can locally be integrated to 2nd order equations, which in turn coincide with the Frønsdal equations in the formulation of [46, 47].

We would like to stress that this is in contrast to previous attempts to construct consistent HS actions [13, 14]: In order to guarantee free field equations of 2nd order, they impose the additional condition that the ‘extra fields’ (in our case the spin-3 connection $\omega_\mu^{ab,cd}$), which are generically of higher derivative order, do not enter the free action. Here we do not have this freedom, since the action is completely determined by gauge invariance, i.e. the extra fields inevitably enter the free theory. That we get nevertheless the correct Frønsdal equations, or in other words, that the higher-derivatives are gauge artefacts that can be eliminated, is due to the curl-like structure of the HS Riemann tensor, which in flat space is known as the Damour-Deser identity [49, 50] (see also [51, 52]). Since we verified here an analogous behaviour on AdS for spin-3, this pattern will most likely extend to all HS fields, and therefore requirement (ii) for consistent HS theories is satisfied.

Let us also stress that in this approach it is very natural, if not necessary, to start with a HS algebra based on trace-full generators, since then the appearance of the compensator α in the ‘integration’ leading from (4.28) to (4.33) has a very natural interpretation in that it compensates for the non-invariance of the pure Frønsdal operator under trace-full HS transformations. Moreover, starting with trace-less generators would imply in particular that the HS Riemann tensor is already trace-less and consequently the field equations in the form (4.27) would be identically satisfied and not lead to any dynamics. Instead, the dynamics could possibly be encoded, via the Bianchi identities, in the lower-rank torsion-like tensors, for which, however, the distinction between constraint equations and dynamical equations would be less straightforward. (See also the discussion about the so-called σ^- -cohomology in [11] and references therein.)

Finally we note that, compared to the ‘unfolded formulation’ of HS theories advertised in the literature so far, the more conventional action principle presented here has the advantage of admitting already a class of exact solutions. In fact, by virtue of the relation (3.23) we concluded that any solution of the purely gravitational theory, as for instance black holes [53] and pp-waves [54], can be lifted to an exact solution of the full theory, simply by setting all HS fields to zero. Accordingly, this theory allows the analysis of HS dynamics on more complicated backgrounds (and then, in principle, also of the back reaction of the geometry). This is in contrast to the unfolded dynamics, for which even in case that all HS fields vanish, the construction of solutions is a highly non-trivial

problem. Indeed, apart from AdS , exact solutions have been found only recently [55, 56].

Many things are left to be done. First, we have analyzed the dynamical content only in case of trivial HS torsion. However, viewed as a 1st order formulation, the theory does not imply vanishing torsion (though the latter provides a particular solution). So either one imposes the torsion constraints by hand, in order to express the deWit–Freedman connections in terms of the physical HS fields, or one treats the torsions as carrying additional degrees of freedom. In the former case it is not clear that this is consistent with the HS gauge symmetry: Although we have seen in sec. 4.1 that in case of a linearization around an AdS geometry the composite connections transform in the same way as the ‘fundamental’ connections – and so imposing the constraints does not violate the HS invariance –, it is not clear whether this is consistent in general. In case it is not consistent, this would mean that there are additional degrees of freedom associated to the torsion, which necessarily need to be taken into account. Apart from that we should point out that due to the way the torsion tensor enters the Chern-Simons theory, there does not exist a 1.5–order formalism.

One of the main difficulties in analyzing the non-linear dynamics of the constructed HS theory in more detail is due to the fact that the infinite-dimensional HS algebras are poorly understood. Though these algebras are well-defined through the oscillator realization described in sec. 2.2, the structure constants, for instance, are not known in general. Further research into this direction is required for a detailed analysis of the interactions.

Once the dynamical content is known, it remains to be seen how the different fields organize into HS multiplets. We first notice that the basic field content of the Chern-Simons theory in $D = 5$ does not fit into multiplets of $\mathfrak{ho}(4, 2)$, since the latter requires in particular a massless scalar [57, 6]. This in turn is the reason that the construction of HS actions à la MacDowell-Mansouri entirely based on a HS gauge field are not believed to be consistent to all orders [14]. However, our case is different, since there are no propagating HS modes around AdS_5 and so there is no reason to expect that the $D = 5$ field content should organize in multiplets.¹⁰ Rather we found that the non-trivial HS dynamics takes place on backgrounds which are not maximally symmetric, as the AdS_4 solution. However, on this background there will most likely be scalar and other excitations which are the Kaluza-Klein modes originating from the off-diagonal components of the various fields. Due to the HS invariance of the full theory, these modes almost by construction will organize into multiplets of $\mathfrak{ho}(3, 2)$, and it would be very interesting to see how this happens. In some sense the theory seems to prevent itself from becoming inconsistent exactly by not having standard dynamics around its most symmetric solution.

Let us finally note that the Chern-Simons theory in $D = 11$ based on (two copies of) the superalgebra $\mathfrak{osp}(1|32)$ has been proposed as the non-perturbative definition of M-theory [61]. As the latter should cover in particular the infinite towers of massive HS states described by 10-dimensional string theory, it is very tempting to conjecture that $\mathfrak{osp}(1|32)$ has to be enhanced to a HS extension, thus giving rise to Chern-Simons actions of the type considered here. In fact, recently it has been argued that the three-dimensional Chern-Simons theory based on a HS algebra is related to M-theory for non-critical strings in $D = 2$ via the background $AdS_2 \times S^1$ [62]. Similarly to the $AdS_4 \times S^1$

¹⁰A similar argument has been employed for supergravity in [58].

solution discussed for the $D = 5$ theory here, one might hope to identify a non-topological 10-dimensional phase, which permits flat Minkowski space and gives rise to massive HS states via spontaneous symmetry breaking.

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A Young Tableaux and Projectors

Here we give a brief review of the technique of Young tableaux used in the main text. As we are exclusively working with trace-full tensors, these encode the irreducible representations of $GL(m)$, as opposed to $SO(m)$ groups. For tensors with AdS_D indices we have $m = D + 1$, while for the corresponding ‘Lorentz’ tensors $m = D$.

A Young tableau consists of a certain number of rows of boxes, where the number of boxes does not increase from top to bottom, as for instance

$$\begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & & \\ \hline \square & \square & \square & & \\ \hline \square & & & & \\ \hline \end{array} . \quad (A.1)$$

It describes the symmetries of an irreducible $GL(m)$ tensor. For the example (A.1) it has the structure $T_{a_1 \dots a_5, b_1 \dots b_3, c_1 \dots c_3, d}$. As a matter of convention we choose the symmetric basis, which means that the corresponding tensors are completely symmetric in all row indices. Specifically, the tensor T above is completely symmetric in the sets of indices $\{a_i\}$, $\{b_i\}$ and $\{c_i\}$, respectively. For irreducibility the tensors have to satisfy the additional condition that symmetrisation of all indices in a certain row with any index corresponding to a box below that row gives zero. For instance, in the example this implies

$$T_{a_1 \dots a_5, (b_1 \dots b_3, c_1) \dots c_3, d} = T_{a_1 \dots a_5, b_1 \dots b_3, (c_1 \dots c_3, d)} = 0, \quad \text{etc.}, \quad (A.2)$$

where ordinary brackets denote complete symmetrization of strength one, as e.g. $T_{(ab)} := \frac{1}{2}(T_{ab} + T_{ba})$. Note that, accordingly, a tensor $T_{a,b}$ in \square is antisymmetric, while in general no specific antisymmetrization properties can be derived from the Young tableau.¹¹ Moreover, one may check that for a tensor in the ‘window’ tableau $\square \square$, eq. (A.2) implies the exchange property $T_{ab,cd} = T_{cd,ab}$.

¹¹It is, however, possible to start with a different convention, in which the *antisymmetrization* properties, i.e. the symmetries in a column of boxes, are specified. In appendix C we have to relate these two.

The language of Young tableaux is efficient in order to determine the decomposition of tensor products into irreducible representations. Specifically, in the tensor product of the ‘vector’ representation \square with any Young tableau, the irreducible parts are obtained by adding \square to the given tableau in all possible ways. For instance, the spin-3 frame field e_μ^{ab} is a priori in the tensor product

$$\square \otimes \square\square = \square\square\square \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}, \quad (\text{A.3})$$

i.e. it contains the completely symmetric (physical) part and the so-called ‘hook’ diagram.

Finally we give the projectors onto the hook and window diagrams, which we need in the main text, explicitly. The hook projector reads on a general tensor with no a priori symmetries

$$(\mathbb{P}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} X)_{abc} \equiv (\mathbb{P}_{(2,1)} X)_{abc} \equiv X_{\langle abc \rangle} = \frac{1}{3} (2X_{(ab)c} - X_{(bc)a} - X_{(ca)b}) . \quad (\text{A.4})$$

Similarly,

$$\begin{aligned} (\mathbb{P}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} X)_{abcd} &\equiv (\mathbb{P}_{(2,2)} X)_{abcd} \equiv X_{\langle abcd \rangle} \\ &= \frac{1}{6} (2X_{(ab)(cd)} + 2X_{(cd)(ab)} - X_{(cb)(ad)} - X_{(ad)(cb)} - X_{(ac)(bd)} - X_{(bd)(ac)}) . \end{aligned} \quad (\text{A.5})$$

Analogous formulas hold in case of different index orderings, as e.g. hook projection according to indices (ab, c) on a tensor X_{cab} ,

$$(\mathbb{P}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} X)_{cab} = \frac{1}{3} (2X_{c(ab)} - X_{a(bc)} - X_{b(ca)}) . \quad (\text{A.6})$$

B Proof of Cyclicity of the Trace

In this appendix we prove the assertion made in sec. 3.2 that the traces Tr_k in (3.21) are cyclic in a general odd dimension $D = 2n - 1$.

For a generic element $F(M)$ in the enveloping algebra $\mathcal{U}[\mathfrak{so}(D-1, 2)]$ the star product with M_{AB} can be computed by use of (3.15),

$$\begin{aligned} M_{AB} \star F &= \left(M_{AB} + 2M_{C[A} \partial^C_{B]} + \frac{2}{3} (M_{CD} \partial_{[A}^C \partial_{B]}^D - M_{C[A} \partial_{B]}^D \partial_D^C) \right) F , \\ F \star M_{AB} &= \left(M_{AB} - 2M_{C[A} \partial^C_{B]} + \frac{2}{3} (M_{CD} \partial_{[A}^C \partial_{B]}^D - M_{C[A} \partial_{B]}^D \partial_D^C) \right) F , \end{aligned} \quad (\text{B.1})$$

which implies

$$[M_{AB}, F(M)]_\star = 4M_{C[A} \partial^C_{B]} F(M) . \quad (\text{B.2})$$

This equation encodes the transformation of $F(M)$ under M_{AB} . In a more mathematical language this states that the BCH star product is strongly $\mathfrak{so}(D-1, 2)$ -invariant [37, 36].

In order to prove the cyclicity of the trace we first show for a *generic* monomial $F_\ell = F^{A_1 B_1 \dots A_\ell B_\ell} M_{A_1 B_1} \dots M_{A_\ell B_\ell}$ of degree ℓ

$$\text{Tr}_1([M_{AB}, F_\ell]_\star) = \text{tr}(\Delta([M_{AB}, F_\ell]_\star)) = 0. \quad (\text{B.3})$$

To see this, we apply Δ to (B.2), whose explicit evaluation gives

$$\begin{aligned} & \text{tr}(\Delta([M_{AB}, F_\ell]_\star)) \\ &= 4\epsilon^{A_1 \dots A_n B_1 \dots B_n} \sum_{r=0}^n \binom{n}{r} \text{tr}(\partial_{A_1 B_1} \dots \partial_{A_r B_r} M_{C[A] \partial_{A_{r+1} B_{r+1}} \dots \partial_{A_n B_n} \partial^C_{|B]} F_\ell) \quad (\text{B.4}) \\ &= \text{tr}([M_{AB}, \Delta F_\ell]_\star) + 4n\epsilon^{A_1 \dots A_n B_1 \dots B_n} \text{tr}(\partial_{A_1 B_1} M_{C[A] \partial_{A_2 B_2} \dots \partial_{A_n B_n} \partial^C_{|B]} F_\ell). \end{aligned}$$

The first term vanishes, which follows from (B.2) and the fact that ‘tr’ sets $M = 0$. The second term can potentially be nonzero when $\ell = n$. In this case it reduces to (after dropping a constant multiplicative factor) $\epsilon^{A_1 \dots A_n B_1 \dots B_{n-1} [A] F_{A_1 B_1 \dots A_{n-1} B_{n-1} A_n | B]}$, which vanishes identically. To see this we use $F^{A_1 B_1 \dots A_\ell B_\ell} = -F^{A_1 B_1 \dots B_m A_m \dots A_\ell B_\ell}$, the symmetry under exchange of any pair (A_m, B_m) and $(A_{m'}, B_{m'})$ and finally the fact that antisymmetrization in $2n + 1$ indices vanishes identically for $\mathfrak{so}(2n)$,

$$\epsilon^{[A_1 \dots A_n B_1 \dots B_{n-1} A] F_{A_1 B_1 \dots A_{n-1} B_{n-1} A_n}{}^B] = 0. \quad (\text{B.5})$$

Furthermore, one can show that for $k \geq 2$

$$\text{Tr}_k([M_{AB}, F_\ell]_\star) = \text{tr}(\Delta^k([M_{AB}, F_\ell]_\star)) = 0, \quad (\text{B.6})$$

by using identities similar to (B.5), which proves that $\text{Tr}_k(F_\ell \star M_{AB})$ is cyclic. By using induction, the proof extends directly to $\text{Tr}_k(F_\ell \star G_{\ell'})$ for an arbitrary monomial $G_{\ell'}$. For this we expand $G_{\ell'} = \sum_m G_m M^m$, use that $M^m = (M)^{\star m} + \sum_{m' < m} c_{m'} M^{\star m'}$ together with associativity, and finally apply (B.6) several times. For instance, when $\ell' = 2$ we find $G_2 = G^{A_1 B_1 A_2 B_2} M_{A_1 B_1} M_{A_2 B_2} = G^{A_1 B_1 A_2 B_2} (M_{A_1 B_1} \star M_{A_2 B_2} - 2\eta_{B_1 A_2} M_{A_1 B_2})$. The cyclicity of the last term follows from the analysis above, and by repeatedly using (B.6) we have that

$$\begin{aligned} G^{A_1 B_1 A_2 B_2} \text{Tr}_k(F_\ell \star M_{A_1 B_1} \star M_{A_2 B_2}) &= G^{A_1 B_1 A_2 B_2} \text{Tr}_k(M_{A_2 B_2} \star F_\ell \star M_{A_1 B_1}) \\ &= G^{A_1 B_1 A_2 B_2} \text{Tr}_k(M_{A_1 B_1} \star M_{A_2 B_2} \star F_\ell). \end{aligned} \quad (\text{B.7})$$

C The Spin-3 Riemann Tensor

Here we summarize some relations for the spin-3 Riemann tensor, most notably the Bianchi identities. (On flat space, a very clear discussion of the spin-3 geometry in metric-like formulation can be found in [49], while aspects of a frame-like formulation are given in [32, 59, 60].) For the proof of the Bianchi identities it will be convenient to work in form language, for which the tensors in (4.6) read

$$\begin{aligned} T^{ab} &= \bar{D}e^{ab} + \omega^{ab,c} \wedge \bar{e}_c, & T^{ab,c} &= \bar{D}\omega^{ab,c} + 3e^{\langle ab} \wedge \bar{e}^c \rangle + \omega^{ab,cd} \wedge \bar{e}_d, \\ R^{ab,cd} &= \bar{D}\omega^{ab,cd} + 4\omega^{\langle ab,c} \wedge \bar{e}^d \rangle. \end{aligned} \quad (\text{C.1})$$

After solving the torsion constraints the Bianchi identity follows by application of \bar{D} to the second torsion tensor,

$$0 = \bar{D}T^{ab,c} = (\bar{D}\omega^{ab,cd} + 4\omega^{(ab,c} \wedge \bar{e}^d) \wedge \bar{e}_d = R^{ab,cd} \wedge \bar{e}_d, \quad (C.2)$$

where we used the first torsion constraint, $T^{ab} = 0$, and the relation

$$\bar{D}^2\omega^{ab,c} = \bar{R}^{ad} \wedge \omega_d^{b,c} + \bar{R}^{bd} \wedge \omega_d^{a,c} + \bar{R}^{cd} \wedge \omega_d^{ab}, \quad (C.3)$$

evaluated for the AdS case (4.16). In components the Bianchi identity (C.2) reads

$$R_{[\mu\nu\rho]\sigma,\lambda\delta} = 0, \quad (C.4)$$

where we converted all indices into curved ones.

These identities can now be used to prove that all traces of the Riemann tensor are algebraically related, or in other words, as in the spin-2 case there is a unique Ricci tensor. First of all, the symmetries of the fiber indices according to the window Young tableau imply $R_{\mu\nu a(b,cd)} = 0$, which in turn shows that

$$R_{\mu\nu ab, c}^c = -2R_{\mu\nu a}^c{}_{,bc} = -2R_{\mu\nu b}^c{}_{,ac}, \quad (C.5)$$

i.e. there is a unique trace in the fiber indices. By virtue of the Bianchi identity (C.4) the trace in the fiber indices can then be related to the trace between one space-time and one fiber index:

$$R_{\mu\lambda}{}^\lambda{}_{\nu,\rho\sigma} = \frac{1}{2}R_{\mu\nu\rho\sigma}{}^\lambda{}_\lambda. \quad (C.6)$$

We are now in a position to rigorously derive the field equations used in the main text. First contracting (4.26) with $g^{\mu\rho}$ yields

$$(D-2)R_{\lambda\delta}{}^\lambda{}_{\nu,\delta}{}^\sigma{}_\sigma = 0, \quad (C.7)$$

where we used (C.4). This in turn implies that the double traces of the Riemann tensor appearing in (4.26) can be set to zero. The remaining terms can be simplified by making repeated use of the Bianchi identity (C.4) and the symmetries of the window tableau:

$$\begin{aligned} 0 &= R_{\mu\lambda}{}^\lambda{}_{\nu,\rho\sigma} - R_{\mu\lambda}{}^\lambda{}_{\sigma,\rho\nu} + (\mu \leftrightarrow \sigma) \\ &= R_{\rho\lambda}{}^\lambda{}_{\nu,\mu\sigma} - R_{\rho\lambda}{}^\lambda{}_{\sigma,\mu\nu} + (\mu \leftrightarrow \sigma) \\ &= 4R_{\rho\lambda}{}^\lambda{}_{\nu,\mu\sigma} + R_{\rho\lambda}{}^\lambda{}_{\mu,\nu\sigma} + R_{\rho\lambda}{}^\lambda{}_{\sigma,\nu\mu} \\ &= 3R_{\rho\lambda}{}^\lambda{}_{\nu,\mu\sigma}. \end{aligned} \quad (C.8)$$

Here we used in the second line the Bianchi identity in μ, λ, ρ , in the third line the window symmetry of the second term in μ, ν, σ and finally the same symmetry in the fourth line. These are the final equations of motion, which basically state that the spin-3 Ricci tensor vanishes. In order to clarify the information contained in this equation, we decompose $R_{\lambda\mu}{}^\lambda{}_{\nu\rho,\sigma}{}^\lambda$ into its irreducible parts. A priori it can take values in the Young tableaux

$$\square \otimes \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array}. \quad (C.9)$$

The origin of these different structures is that in the frame-like formulation the Riemann tensor necessarily appears in a mixed basis in the sense that the antisymmetric 2-form indices are on a different footing as the frame indices. To compare with the completely symmetric or completely antisymmetric basis used in the metric-like formulation in [30] and [49], respectively, we have to impose these symmetries, i.e. we define

$$\mathcal{R}_{\mu\nu;\rho\sigma;\lambda\delta}^{(a)} = R_{\mu\rho\nu\lambda,\delta\sigma} \quad , \quad \mathcal{R}_{\mu\nu\rho,\sigma\lambda\delta}^{(s)} = R_{(\mu}{}^{(\sigma}{}_{\nu\rho),\sigma}{}^{\lambda\delta)} \quad . \quad (\text{C.10})$$

Since these are in definite Young tableaux (namely both in $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$, depending on the chosen conventions for symmetrisation or antisymmetrisation properties), it is easily seen that there is a unique trace. Explicitly one finds

$$\mathcal{R}_{\mu\nu;\rho;\sigma}^{(a)} = \frac{3}{8} R_{\mu\nu\rho\sigma}{}^{\lambda}{}_{\lambda} \quad , \quad \mathcal{R}_{\mu\nu\rho,\sigma}^{(s)} = \frac{1}{2} R_{\lambda(\mu}{}^{\nu\rho),\sigma}{}^{\lambda}{}_{\lambda} \quad . \quad (\text{C.11})$$

With these relations it follows that this trace of $\mathcal{R}^{(s)}$ is in $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$, while its algebraically related trace is in $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$. Similarly, the trace of $\mathcal{R}^{(a)}$ takes values in $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$, but interpreted in the antisymmetric basis. To summarize, taking the trace in the fiber indices of the Riemann tensor in the mixed basis corresponds to the Ricci tensor in the completely antisymmetric basis, while a trace between space-time and fiber index corresponds to the Ricci tensor in the completely symmetric basis.

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